Nonoptimality of the Steady-State Cruise for Aircraft

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For a fairly general aircraft model and a large class of drag models, steady-state cruise for a long time span is nonoptimal with respect to fuel economy. This is proved by a second-order variational analysis, using a frequency-domain version of the classical Jacobi (conjugate point) optimality condition. The variational analysis suggests a sinusoidal perturbation away from steady-state cruise which improves fuel economy (as confirmed numerically), but is still not optimal. The form of the optimal trajectory for long duration cruise is unknown. However, two intuitive reasons for improved fuel economy using cycle cruise paths are given.

I. Introduction

In a paper by Schultz and Zagalsky[1] various simplified aircraft dynamic models are discussed with respect to minimum fuel trajectories. For the simplest, the energy state approximation, it is clearly seen through the Legendre-Clebsch condition that intermediate values of thrust can never be minimizing. Therefore, cruise is not a sustainable extremal path. By adding the differential equation for altitude, and using flight path angle and thrust as the control variables, Ref. 1 shows that the Euler-Lagrange equations are satisfied along the cruise arc. However, as shown in Ref. 2, the generalized Legendre-Clebsch condition (or Kelley condition)[3] is not satisfied, and again the cruise arc is shown not to be minimizing. In response to Ref. 2, Schultz[4] considered a higher order set of equations by making flight path angle a state variable, and using lift and thrust as the control variables. Schultz shows that both the first-order necessary conditions and the generalized Legendre-Clebsch condition are satisfied at the cruise point.

The question is where did this previous lack of convexity in the controls go when considering the higher order dynamic equations in Ref. 4? The final and most difficult test to apply is the Jacobi test or conjugate point test.[5] This test is difficult because it does not test the extremal path at a point, but over a finite time interval. Fortunately, since the cruise arc is static (the mass change is assumed negligible), the second variational problem is time invariant, and lends itself to frequency-domain techniques. By considering an infinite time problem with both initial and terminal constraints an *algebraic* test is obtained which gives both a necessary and sufficient condition for the second variation of the cost to be negative.[6] The cruise arc for the given dynamic system is a singular arc[3,5] with respect to the thrust control. Therefore, using Goh's transformation[7] and another algebraic transformation on the control variables, the frequency test is reduced to a scalar condition. For the models considered the nonoptimality of the cruise arc is established using this frequency test and, therefore, a conjugate point occurs. Although we do not know precisely how long an arc must be traversed before a conjugate point occurs, the frequency nature of this test gives some indication. It should be clear that this paper does *not* report on what is the optimizing arc, but only on the optimality of the cruise arc.

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In Sec. II, the problem is defined and the optimization problem discussed. In Sec. III the second variation about the cruise arc is developed by using the Goh transformation.[7] The frequency-domain inequality is presented in Sec. IIIC which is a necessary and sufficient condition for the second variation of the cost to be negative. In Sec. IV the feedback form of the oscillatory control is presented. In Sec. V two examples of sufficient generality are given which will help give insight into the theoretical work. For example, the nonminimizing oscillatory control sequence, suggested by the second variation problem in the frequency domain, is shown in a numerical example to give a smaller fuel cost than that of the cruise arc. Finally, in Sec. VI two reasons why cyclic operation improves fuel economy are given.

II. Dynamic Optimization Problem

Consider the dynamic equations for an airplane of the form

$$\dot{V} = [(T-D)/M]g\sin\gamma \tag{1}$$

$$\dot{\gamma} = (L - W\cos\gamma)/MV \tag{2}$$

$$\dot{h} = V\sin\gamma \tag{3}$$

$$\dot{x} = V\cos\gamma \tag{4}$$

where V is velocity, h is altitude, γ is flight path angle, x is down range, M is the mass assumed constant, L is the lift force, T is the thrust force bounded by the inequality constraint $T_{\min}(V,h) \le T \le T_{\max}(V,h)$, D is the drag force assumed to be a function of (h,V,L), W is the weight, and g is the gravitational acceleration. The problem is to minimize the fuel used, denoted as

$$J = \int_0^{t_f} \sigma T \mathrm{d}t, \tag{5}$$

subject to the dynamic constraints Eqs. (1) to (4). The specific fuel consumption σ is assumed constant in Secs. II to V in order to simplify the analysis. However, in Sec. VI results are given for a case where σ is a function of velocity.

It should be pointed out that the linear form for the thrust in the integral of Eq. (5) is sometimes due to relaxing a concave function of thrust versus mass rate, and thereby avoiding mathematically infinite chattering solutions. Note that if cruise is not minimizing, then these chattering solutions may well be avoided. This should translate into additional savings since relaxed intermediate thrusting arcs are not realizable.

A. Static Optimization: The Cruise Conditions

The above optimization problem can be greatly simplified if a static optimization problem is solved. If the independent variable is changed from t to x, then using Eqs. (4) the integrand in Eq. (5) becomes $\sigma T/V$. In the static optimization problem $\dot{V}=\dot{\gamma}=\dot{h}=0$ which implies that along the cruise arc from Eqs. (1) to (3)

$$T=D, \quad L=W, \quad \gamma=0 \tag{6}$$

The best altitude and velocity are determined by minimizing $(\sigma D/V)$ with respect to V and h. The object of the following is to test the optimality of the static path using the necessary and sufficient conditions of the variational calculus. The variational Hamilton is defined as

$$H \triangleq \sigma T+\lambda_V\left\{\frac{T-D}{M}-g\sin\gamma\right\}+\lambda_\gamma\{(L-W\cos\gamma)/MV\}$$

$$+\lambda_h(V\sin\gamma)+\lambda_x V\cos\gamma \tag{7}$$

where λ_V, λ_γ, λ_h, and λ_x are Lagrange multipliers associated with the dynamic Eqs. (1) to (4), respectively. The Euler-Lagrange equations† are

$$-\dot{\lambda}_V=H_V=-(D_V/M)\lambda_V-\lambda_\gamma(L-W\cos\gamma)/MV^2$$

$$+\lambda_h\sin\gamma+\lambda_x\cos\gamma \tag{8}$$

$$-\dot{\lambda}_\gamma=H_\gamma=-\lambda_V g\cos\gamma+\lambda_\gamma W\sin\gamma/MV+\lambda_h V\cos\gamma-\lambda_x V\sin\gamma \tag{9}$$

$$-\dot{\lambda}_h=H_h=-\lambda_V D_h/M \tag{10}$$

$$-\dot{\lambda}_x=0 \tag{11}$$

and the optimality conditions are

$$H_T=\sigma+\lambda_V/M=0 \tag{12}$$

$$H_L=-\lambda_V D_L/M+\lambda_\gamma/MV=0 \tag{13}$$

Furthermore, since time is unconstrained, $H=0$. It is assumed here that $T=D$ lies in the interior of the region of admissible controls. Since $H_{TT}\equiv 0$ and $H_{TL}\equiv 0$, the cruise arc is a singular arc in the calculus of variations.[3,5] For a particular drag model, Schultz[4] showed that the generalized Legendre-Clebsch condition[3] is satisfied. Since the path is static then the conditions for the cruise arc must imply that $\dot{\lambda}_V=\dot{\lambda}_\gamma=\dot{\lambda}_h=0$. From the stationary condition $(\sigma D/V)_h=0$, $\lambda_h=0$ is obtained. From Eq. (12) and from $H=0$ we determine (variables evaluated along the cruise arc are denoted by a subscript c)

$$\lambda_V=-\sigma M, \quad \lambda_x=-\sigma D_c/V_c \tag{14}$$

From the stationary condition $(\sigma D/V)_V=0$

$$-\dot{\lambda}_V=+\left(\frac{\sigma D_V}{V}-\frac{\sigma D}{V^2}\right)_c V_c=0 \tag{15}$$

Using the optimality condition Eq. (13) and λ_V given in Eq. (14),

$$\lambda_\gamma=-\sigma M(D_L)_c V_c \tag{16}$$

Finally, since λ_γ is a constant along the cruise arc, Eq. (9) implies that

$$\lambda_h=-\sigma W/V_c \tag{17}$$

The first-order necessary conditions are seen to be satisfied along the cruise arc and the Lagrange multipliers are easily evaluated.

III. Second-Order Tests of Optimality

By expanding the performance index to second-order, and the dynamics to first order about the cruise arc, the accessory problem in the calculus of variations is formed which will establish the nonoptimality of the static cruise. Define (δV, $\delta\gamma$, δh, δx, δT, δL) as small variations in (V, γ, h, x, T, L) away from their respective cruise values. For notational simplicity, define the vectors of variations in the state and control variables as

$$x^T \triangleq [\delta V,\delta\gamma,\delta h,\delta x], \quad y^T \triangleq [\delta T,\delta L] \tag{18}$$

The accessory problem is that of finding a control sequence $y(t)$, $t\epsilon[0,t_f]$, which minimizes

$$2\delta^2 J=\int_{t_0}^{t_f}[x^T y^T]\begin{bmatrix} Q & C \\ C^T & R\end{bmatrix}\begin{bmatrix} x \\ \hline y\end{bmatrix}\mathrm{d}\tau \tag{19}$$

subject to the linearized dynamic Eqs.

$$\dot{x}=Ax+By \tag{20}$$

with boundary conditions

$$x(0)=x(t_f)=0 \tag{21}$$

where the matrices involved in Eq. (19) and (20) are

$$Q\triangleq\begin{bmatrix} H_{VV} & H_{V\gamma} & H_{Vh} & 0 \\ H_{V\gamma} & H_{\gamma\gamma} & 0 & 0 \\ H_{Vh} & 0 & H_{hh} & 0 \\ 0 & 0 & 0 & 0\end{bmatrix}, \quad C\triangleq\begin{bmatrix} 0 \\ C_L\end{bmatrix}\triangleq\begin{bmatrix} 0 & 0 & 0 & 0 \\ H_{VL} & 0 & H_{Lh} & 0\end{bmatrix}, \quad R\triangleq\begin{bmatrix} 0 & 0 \\ 0 & H_{LL}\end{bmatrix} \tag{22a}$$

$$A\triangleq\begin{bmatrix} -D_V/M & -g & -D_h/M & 0 \\ 0 & 0 & 0 & 0 \\ 0 & V & 0 & 0 \\ 1 & 0 & 0 & 0\end{bmatrix}, \quad B\triangleq[B_T,B_L]\triangleq\begin{bmatrix} 1/M & -D_L/M \\ 0 & 1/MV \\ 0 & 0 \\ 0 & 0\end{bmatrix} \tag{22b}$$

†Unless otherwise defined the partial derivative of some scalar function $f(x)$ is denoted $f_x=\partial f/\partial x$ and $f_{xx}=\partial^2 f/\partial x^2$.

The second partial of the Hamiltonian involved in the Q, C, and R matrices of Eq. (22) are

$$H_{VV}=-\frac{D_{VV}}{M}\lambda_V,\ H_{V\gamma}=\lambda_h,\ H_{Vh}=-\frac{D_{Vh}}{M}\lambda_V,\ H_{Vx}=0 \quad (23a)$$

$$H_{\gamma\gamma}=\lambda_\gamma W\cos\gamma/MV-\lambda_x V\cos\gamma,\ H_{\gamma h}=0,\ H_{\gamma x}=0 \quad (23b)$$

$$H_{hh}=-\lambda_V D_{hh}/M,\ H_{hx}=0,\ H_{xx}=0,\ H_{TT}=0,\ H_{TL}=0 \quad (23c)$$

$$H_{VL}=-\frac{D_{VL}}{M}\lambda_V-\lambda_\gamma/MV^2,\ H_{hL}=-\lambda_V D_{hL}/M,$$

$$H_{LL}=-\lambda_V D_{LL}/M \quad (23d)$$

Note that the first and second variations of the cost with respect to variations in the terminal time are zero.

A. Goh's Transformation and the Generalized Legendre-Clebsch Condition

The two second-order necessary conditions to be used as tests are the Legendre-Clebsch condition and the Jacobi condition. Since at present the Legendre-Clebsch condition can only be found in weak form (R is semipositive definite), a transformation of control and state variables is suggested to obtain a generalized Legendre-Clebsch condition in strong form. Also using this transformation, the Jacobi test will also be made. The following transformation, due to Goh,[3,7] is: Define

$$v\triangleq\int_{t_0}^{t}\delta T(\tau)\,\mathrm{d}\tau \quad (24)$$

and

$$z\triangleq x-B_T v,\quad x=z+B_T v \quad (25)$$

where v is a new control variable replacing δT and z is a new state vector replacing x. Substituting Eq. (25) into (20), the dynamic equation in the new variables is

$$\dot z=\dot x-B_T\dot v=Az+[AB_T,B_L]\left[\begin{smallmatrix}v\\ \delta L\end{smallmatrix}\right] \quad (26)$$

The new accessory minimum problem is to find $v(t)$ and δL which minimizes the performance index Eq. (19) in the transformed variables given as

$$2\delta^2 J=\int_{t_0}^{t_f}\left\{z^TQz+2[v,\delta L]\begin{bmatrix}B_T^TQ\\ C_L\end{bmatrix}z+[v,\delta L]\begin{bmatrix}B_T^TQB_T & C_LB_T\\ B_T^TC_L^T & H_{LL}\end{bmatrix}\begin{bmatrix}v\\ \delta L\end{bmatrix}\right\}\mathrm{d}\tau \quad (27)$$

subject to Eq. (26) with boundary conditions

$$z(0)+B_Tv(0)=z(t_f)+B_Tv(t_f)=0 \quad (28)$$

Note the nonstandard form of the boundary conditions Eq. (28) where the control variable enters the terminal constraint. For additional detail see Refs. 3 and 8. From Eq. (27) the generalized Legendre-Clebsch condition is

$$R'\triangleq\begin{bmatrix}B_T^TQB_T & C_LB_T\\ B_T^TC_L^T & H_{LL}\end{bmatrix}=\begin{bmatrix}H_{VV}/M^2 & H_{VL}/M\\ H_{VL}/M & H_{LL}\end{bmatrix}\geq 0 \quad (29)$$

B. A Transformation of Control Variables

An additional transformation (c.f. Chapt. 5, Problem 4 in Ref. 5) is made which will remove the cross term between control and state variables in Eq. (27). Furthermore, it will be seen that the dimension of the state space in the second variational problem can be reduced to three. If we define

$$N^T\triangleq\begin{bmatrix}B_T^TQ\\ C_L\end{bmatrix},\quad u\triangleq\begin{bmatrix}v\\ \delta L\end{bmatrix} \quad (30)$$

then an equivalent problem to Eqs. (26) to (28) is to minimize

$$2\delta^2J=\int_{t_0}^{t_f}\{z^T[Q-N(R')^{-1}N^T]z+\bar u^TR'\bar u\}\mathrm{d}\tau \quad (31)$$

subject to

$$\dot z=\{A-[AB_T,B_L](R')^{-1}N^T\}z+[AB_T,B_L]\bar u \quad (32)$$

with boundary

$$z(0)+B_Tv(0)=z(t_f)+B_Tv(t_f)=0 \quad (33)$$

where the control $\bar u$ contains a feedback term as

$$u=\bar u-(R')^{-1}N^Tz;\quad \bar u^T\triangleq[\bar v,\delta\bar L] \quad (34)$$

Define

$$K\triangleq H_{VV}H_{LL}-H_{VL}^2 \quad (35)$$

Then

$$[Q-N(R')^{-1}N^T]\triangleq Q'=$$

$$\begin{bmatrix}0 & 0 & 0 & 0\\ 0 & H_{\gamma\gamma}-H_{LL}H_{V\gamma}^2/K & -H_{V\gamma}(H_{LL}H_{Vh}-H_{VL}H_{Lh})/K & 0\\ 0 & -H_{V\gamma}(H_{LL}H_{Vh}-H_{VL}H_{Lh})/K & H_{hh}-(H_{LL}H_{Vh}^2-2H_{Lh}H_{VL}H_{Vh}+H_{VV}H_{Lh}^2)/K & 0\\ 0 & 0 & 0 & 0\end{bmatrix} \quad (36)$$

and

$$[A-[AB_T,B_L](R')^{-1}N^T]\triangleq A'=$$

$$\begin{bmatrix}0 & \left\{-g-\frac{H_{V\gamma}}{MK}(D_LH_{VL}-H_{LL}D_V)\right\} & \left\{-\frac{D_h}{M}+\frac{D_V}{MK}(H_{LL}H_{Vh}-H_{VL}H_{Lh})+\frac{D_L}{MK}(H_{VV}H_{Lh}-H_{VL}H_{Vh})\right\} & 0\\ 0 & H_{VL}H_{V\gamma}/MVK & -(H_{VV}H_{Lh}-H_{VL}H_{Vh})/MVK & 0\\ 0 & V & 0 & 0\\ 0 & -H_{LL}H_{V\gamma}/K & -(H_{LL}H_{Vh}-H_{VL}H_{Lh})/K & 0\end{bmatrix} \quad (37)$$

By making this transformation, the performance index and the dynamics do *not* depend upon variations z_1 or z_4. This is because the matrix $[Q-N(R')^{-1}N^T]$ is a projector which annihilates any vector in the velocity direction (e.g., multiply this matrix by B_T). Note that since z_1 does not effect (z_2,z_3,z_4) in the dynamics Eq. (32), does not enter the performance index Eq. (31), and is unconstrained at the boundaries Eq. (33), z_1 can be eliminated from the problem. The boundary conditions Eq. (33) are discussed in Sec. IV.

The elimination of z_1 from the problem, which reduces the dimension of the state space by one, is due to the singularity of the cruise arc. This is consistent with the results in Ref. 8 where the Kelley transformation is used directly to reduce the dimension of the state space along a singular arc. Since δT enters the accessory problem only through the differential equation for the velocity variation, Goh's transformation, when σ is assumed constant, is equivalent to considering δV as the new control variable. However, when considering a general function for σ, Goh's transformation gives a general method for obtaining the generalized Legendre-Clebsch condition. Referring back to Goh's transformation Eq. (25), note that $z_2=\delta\gamma$, $z_3=\delta h$ and $z_4=\delta x$.

C. A Frequency Domain Test for the Positivity of $\delta^2 J$

Since the cruise arc is stationary, the accessory problem is a stationary linear quadratic problem of infinite duration. By using Parseval's relation we can transform from the time domain to the frequency domain.[6] Since z_1 is not of interest, define the state vector as $\bar{z}^T=(z_2,z_3,z_4)$. The second variational problem is to minimize

$$2\delta^2 J=\int_0^\infty (\bar{z}^T Q\bar{z}+\bar{u}^T R'\bar{u})\,\mathrm{d}\tau \tag{38}$$

subject to

$$\dot{\bar{z}}=A\bar{z}+B\bar{u},\quad \bar{z}(0)=\bar{z}(\infty)=0 \tag{39}$$

where

$$Q\triangleq\begin{bmatrix} Q'_{22} & Q'_{23} & 0\\ Q'_{32} & Q'_{33} & 0\\ 0 & 0 & 0\end{bmatrix} \tag{40}$$

$$A\triangleq\begin{bmatrix} A'_{22} & A'_{23} & 0\\ A'_{32} & 0 & 0\\ A'_{42} & A'_{43} & 0\end{bmatrix} \tag{41}$$

$$B\triangleq\begin{bmatrix} 0 & 1/MV\\ 0 & 0\\ 1/M & 0\end{bmatrix} \tag{42}$$

where the elements in Eqs. (40) and (41) are defined in Eqs. (36) and (37), and where B is the second to fourth row of $[AB_T,B_L]$ given in Eq. (32).

Using Parseval's relation in Eq. (38) we have

$$2\delta^2 J=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}[\bar{z}(-i\omega)^T Q\bar{z}(i\omega)+\bar{u}(-i\omega)^T R'\bar{u}(i\omega)]\,\mathrm{d}\omega \tag{43}$$

By transforming Eq. (39) as

$$\bar{z}(i\omega)=[I(i\omega)-A]^{-1}B\bar{u}(i\omega) \tag{44}$$

Eq. (43) becomes

$$2\delta^2 J=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\bar{u}(-i\omega)^T\{B^T(I(-i\omega)-A^T)^{-1}\times Q(I(i\omega)-A)^{-1}B+R'\}\bar{u}(i\omega)\mathrm{d}\omega \tag{45}$$

Theorem: If the system $\dot{\bar{z}}=A\bar{z}+B\bar{u}$ is controllable, then a necessary and sufficient condition that $\delta^2 J$ be negative for the accessory minimum problem posed in Eq. (38) and (39) is that there exists some frequency ω such that

$$Y(\omega)=B^T[I(-i\omega)-A^T]^{-1}Q[I(i\omega)-A]^{-1}B+R' \tag{46}$$

is *not* nonnegative definite.

Proof: The proof is given by Willems[6] [Theorem 1 and 4] where he proves that a necessary and sufficient condition for $\delta^2 J$ to be nonnegative definite is that $Y(\omega)$ is nonnegative definite for all ω. Since we are interested in the converse problem, the proofs of necessity and sufficiency are reversed.

Remark 1: If $Y(\omega)$ is not nonnegative definite for some ω then the extremizing cruise arc is not minimizing.

Remark 2: Note that by using purely oscillatory terms, the boundary conditions are satisfied at $t_f\to\infty$. The case of fixed terminal boundary condition is the only case for which Willems results are, in general, both necessary and sufficient.[9] Furthermore, a result similar to the above theorem but again based upon Willems work is given in Ref. 10.

Using Eqs. (40) to (42), Eq. (46) becomes

$$Y(\omega)=\begin{bmatrix}\dfrac{H_{VV}}{M^2} & H_{VL}/M\\ H_{VL}/M & H_{LL}+\dfrac{1}{M^2V^2}\left(\dfrac{Q'_{22}\omega^2+Q'_{33}A'^2_{32}}{(\omega^2+A'_{23}A'_{32})^2+A'^2_{22}\omega^2}\right)\end{bmatrix} \tag{47}$$

This result can be further reduced by noting that since z_4 does not influence $\delta^2 J$ in Eq. (37) minimizing $\delta^2 J$ with respect to $\bar{v}$ gives the relation

$$\bar{v}=-(MH_{VL}/H_{VV})\,\delta\bar{L} \tag{48}$$

If the relation Eq. (48) is used in Eq. (45), $Y(\omega)$ becomes the scalar function

$$Y(\omega)=K/H_{VV}+1/M^2V^2\left(\frac{Q'_{22}\omega^2+Q'_{33}A'^2_{32}}{(\omega^2+A'_{23}A'_{32})^2+A'^2_{22}\omega^2}\right) \tag{49}$$

The central result is the condition that if $Y(\omega)$ given by Eq. (49) is negative for some ω, then the cruise arc is nonminimizing.

IV. The Form of the Variational Oscillatory Controls

Going back through the transformations, the form of δT and δL needed to produce a neighboring control with lower cost than the extremal path is determined. The gain in Eq. (34) is

$$(R')^{-1}N^T=\begin{bmatrix} M & MH_{LL}H_{V\gamma}/K & M[H_{LL}H_{Vh}-H_{Lh}H_{VL}]/K & 0\\ 0 & -H_{V\gamma}H_{VL}/K & [H_{VV}H_{Lh}-H_{Vh}H_{VL}]/K & 0\end{bmatrix} \tag{50}$$

Noting that

$$z_1=\delta V-v/M,\ z_2=\delta\gamma,\ z_3=\delta h,\ z_4=\delta x,$$

Eq. (34) using Eq. (50) becomes

$$0=\dot{v}-M\delta V-\frac{MH_{LL}H_{V\gamma}}{K}\delta\gamma-\frac{M}{K}[H_{LL}H_{Vh}-H_{Lh}H_{VL}]\delta h \tag{51}$$

$$\delta L=\frac{H_{V\gamma}H_{VL}}{K}\delta\gamma-\frac{[H_{VV}H_{Lh}-H_{Vh}H_{VL}]}{K}\delta h+\delta\bar{L} \tag{52}$$

Note that in Eq. (51) v cancels out. Using Eq. (48), a relationship between δV and $\delta\gamma$, δh, and δL results from Eq. (51) as

$$\delta V=-(H_{LL}H_{V\gamma}/K)\delta\gamma - ([H_{LL}H_{Vh}-H_{Lh}H_{VL}]/K)\delta h-(H_{VL}/H_{VV})\delta\bar{L} \tag{53}$$

In applying the boundary conditions using the previous conditions, we do not insist that the small variations $x(0)=x(\infty)$ be zero. The effect on the cost in going from zero to $x(0)$ and from $x(\infty)$ to zero is negligible compared to the cost over the infinite time interval.

If Eq. (53) is differentiated the variation in the thrust is

$$\delta T=D_V\delta V+[Mg-MV(H_{LL}H_{Vh}-H_{Lh}H_{VL})/K]\delta\gamma+D_h\delta h + [D_L-H_{LL}H_{V\gamma}/KV]\delta L-H_{VL}/H_{VV}\dot{\delta\bar{L}} \tag{54}$$

where $\delta\bar{L}$ is differentiable because it is a purely oscillatory function. The perturbing controls δL and δT, given by Eqs. (52) and (54), are complicated by being composed of an oscillatory term $\delta\bar{L}$ and feedback terms.

V. Specialization to Particular Drag Models

Consider the drag model of the form

$$D=C_{D_0}(V)qS+\zeta(V)L^2/qS \tag{55}$$

where $C_{D_0}(V)$ is the zero-lift drag coefficient, $\zeta(V)$ is the induced drag factor, S is the surface area and q is the dynamic pressure

$$q=\rho V^2/2 \tag{56}$$

If we restrict our problem to the stratosphere where an isothermal atmosphere is assumed, then the atmospheric density ρ becomes an exponential function of altitude

$$\rho=\rho_0 e^{-\beta h} \tag{57}$$

where ρ_0 and β are constants and the speed of sound becomes a constant.

It should be noted that along the cruise arc using the drag model Eq. (55)

$$\left(\frac{\sigma D}{V}\right)_h=0\Rightarrow C_{D_0}qS=\zeta(V)W^2/qS=D/2 \tag{58}$$

Using the identifications in Eq. (23) along with Eqs. (55) to (58), then

$$\begin{gathered} H_{LL}=\sigma D/W^2,\ H_{V\gamma}=\lambda_h=-\sigma W/V,\ H_{Vh}=\sigma D_{Vh} \\ H_{hL}=\sigma D\beta/W,\ H_{VV}=\sigma D_{VV}, \\ H_{hh}=\sigma D\beta^2,\ H_{VL}=\sigma(D_{VL}+D_L/V) \\ K=\sigma^2[D_{VV}D_{LL}-(D_{LV}+D_L/V)^2]>0,\ H_{\gamma\gamma}=0 \end{gathered} \tag{59}$$

Note that for drag models of the type (55), $H_{\gamma\gamma}=0$. This means that Q in Eq. (38) is *not* nonnegative definite (c.f. (40)) and the possibility of a conjugate point exists.

A. A Further Specialization of the Drag Model (Case 1)

Assume that

$$\zeta(V)=\psi V \tag{60}$$

where ψ is a given constant. Using Eq. (59) the simplification results

$$H_{VL}=0 \tag{61}$$

As a result of Eq. (61), $Y(\omega)$ given by Eq. (49) becomes

$$Y(\omega)=H_{LL}-\sigma[\omega^2W^2/V^2+V^2D_{Vh}^2]/M^2V^2D_{VV}(\omega^2-\beta g)^2 \tag{62}$$

In this case both Q'_{22} and Q'_{33} defined in Eq. (36) and used in Eq. (38) are negative.

The sign of $Y(\omega)$ is investigated for various values of ω. If $\omega\to\infty$, $Y(\omega)\to H_{LL}>0$. Infinite chattering solutions are not optimum. If $\omega\to 0$, then

$$Y(0)=\sigma(D_{LL}-D_{hV}^2D_{LL}^2/D_{VV}D_{Lh}^2)$$

However, if the determinate of the second variation of the static minimization of $(\sigma D/V)$ is set to zero, then

$$D_{VV}=D_{hV}^2/D_{hh},\quad\text{and}\quad Y(0)=0$$

using Eq. (59). If the determinate is positive then $Y(0)>0$. Therefore, for low frequencies $Y(\omega)$ remains nonnegative.

If $\omega\to\sqrt{\beta g}$,‡ then $Y(\omega)\to-\infty$. By choosing $\delta\bar{L}$ to oscillate at this frequency the cost δ^2J is made negative. About $\omega=\sqrt{\beta g}$ there is a range of frequencies for which $Y(\omega)$ is negative. Note that a frequency of $\sqrt{\beta g}$ has a period of approximately 2.8 min, and it would be anticipated that a conjugate point would occur at time intervals of about this magnitude. If the time of the conjugate point was of interest the associated Riccati equation[5] could be integrated. The escape time of the Riccati equation is where the conjugate point occurs.

B. Numerical Verification

The equations of motion Eqs. (1) to (4) along with the performance index Eq. (5) were programed on a digital computer. The object is to show numerically that if the control laws given by Eqs. (52) and (54) were mechanized in the nonlinear equations of motion, then the actual cost should be reduced over that of the cruise arc. However, this will be a second-order change in the fuel cost.

In particular, choose the drag coefficient as

$$C_{D_0}(V)=P\{[(V-V_0)^2+\epsilon]^{1/2}+(V-V_0)\}^2+C_{D_0}(0) \tag{63}$$

where $V_0=1000$fps, $\epsilon=1(\text{fps})^2$, $C_{D0}(0)=.02$ and $P=1$sec/ft. The scale height $1/\beta=2.35\times10^4$ft, $\rho_0=2.7\times10^{-3}$ slugs/ft^3, $S=3000$ ft, $\psi=.001$ sec/ft, $\sigma=10^{-3}$ sec^{-1}, $M=6000$ slugs and $g=32.2$ fps. No attempt was made to represent a particular aircraft.

The cruise arc using the above data occurred at $V_c=971.2$ fps and $h_c=24{,}683.7$ ft where the subscript c denotes conditions calculated on the cruise arc. The matrix associated with the second variation of the static minimization of $(\sigma D/V)$ is positive definite.

The forcing control $\delta\bar{L}$ is of the form

$$\delta\bar{L}=W\text{Weig}(b\cos\sqrt{\beta g}\,t+\sin\sqrt{\beta g}\,t) \tag{64}$$

‡Consider an alternate interpretation of the frequency $\sqrt{\beta g}$. If it is assumed that 1) the velocity is constant, 2) the lift is a function of altitude such that $L_h=-L\beta$, and 3) along the cruise arc $L=W$, then the variational equations of (2) and (3) result in $\ddot{\delta h}+g\beta\delta h=0$.

Table 1 Linear performance of nonlinear simulation

$\gamma(0)$ rad	$\delta V(0)$ fps	ΔV fps	$\Delta\gamma$ rad	Δh ft	$(\mathrm{d}J/x)_{\mathrm{actual}}$ lb/ft	$(\mathrm{d}J/x)_{\mathrm{pred}}$ lb/ft	$\delta^2 J/x$ lb/ft
				Four cycles calculated			
10^{-4}	6.34×10^{-3}	7.78×10^{-6}	-1.82×10^{-10}	-1.5×10^{-6}	-1.149×10^{-10}	-1.149×10^{-10}	-4.53×10^{-11}
10^{-3}	6.34×10^{-2}	7.79×10^{-4}	-1.81×10^{-8}	-9.2×10^{-5}	-1.151×10^{-8}	-1.151×10^{-8}	-4.53×10^{-9}
10^{-2}	6.34×10^{-1}	7.84×10^{-2}	-1.78×10^{-6}	4.9×10^{-2}	-1.173×10^{-6}	-1.171×10^{-6}	-4.49×10^{-7}
				One cycle calculated			
10^{-1}	6.34	2.14	1.02×10^{-5}	15.5	-1.414×10^{-4}	-1.396×10^{-4}	-4.29×10^{-5}

[a] $\mathrm{Weig}/\gamma(0)=10^{-2}$

where b and the stopping time are chosen so that the initial and terminal boundary conditions on the undamped oscillator of Eq. (41) using only z_2 and z_3 are matched from linear theory. Weig is chosen small to insure the validity of the linearization assumption.

The equations of motion were integrated over 4 cycles. At the stopping time the fuel consumed was normalized by dividing by the distance travelled on the oscillatory trajectory $(\mathrm{fuel}/x)_{\mathrm{osc}}$. By subtracting this from the normalized cost along the cruise arc, $(\mathrm{fuel}/x)_{\mathrm{cruise}}=\sigma T_c/V_c$, the actual normalized change in cost is defined as

$$(dJ/x)_{\mathrm{actual}}=(\mathrm{fuel}/x)_{\mathrm{osc}}-\sigma T_c/V_c \tag{65}$$

The resulting value was compared with the predicted normalized change in cost expanded up to second-order as

$$(dJ/x)_{\mathrm{pred.}}=[\lambda_V\Delta V+\lambda_\gamma\Delta\gamma+\lambda_h\Delta h+\delta^2 J]/x \tag{66}$$

where $\lambda_V,\lambda_\gamma,\lambda_h$ are given in Eqs. (14), (16), and (17) and $\delta^2 J$ is given in Eq. (19); and whereas the second-order effects cause the initial and terminal boundary conditions not to match, the following variations in Eq. (66) are defined as

$$\Delta V=V(0)-V(t_f),\Delta\gamma=\gamma(0)-\gamma(t_f),\Delta h=h(0)-h(t_f) \tag{67}$$

The greatest violations were found in velocity. Table 1 shows how well the linearity is held as the variation in initial γ is changed by four orders of magnitude. The initial variation in altitude and range is zero, but the initial variation in velocity is related to the initial variation in γ by (53) where for this problem $H_{VL}=0$. The cost in going from the cruise arc to these initial conditions, and then from the final conditions back to the cruise arc is negligible compared to the cost savings in going over numerous cycles. Note that the changes in ΔV and $\Delta\gamma$ change by the same orders of magnitude as the second-order term $\delta^2 J/x$. The predicted and actual values of $(\mathrm{d}J/x)$ remain quite close although there is noticeable degeneration as $\gamma(0)$ goes to 0.1 rad. Since the predicted and actual values of $(\mathrm{d}J/x)$ match to second-order, the term $\delta^2 J/x$ is approximately the cost savings if the initial and terminal boundaries were perfectly matched. In the case where $\gamma(0)=0.1$ rad only one cycle was calculated because over 4 cycles the linearity assumption was violated. This case gave about 0.08% improvement in fuel without violating the linearity assumption very badly. The $(L/D)_{\max}$ for the assumed lift-drag polar at cruise is less than 4. If $\psi=0.0001$ sec/ft, then the $(L/D)_{\max}$ at cruise goes to about 10 and the fuel improvement for $\gamma(0)=0.1$ rad is 0.4%. The optimal paths are unknown.

C. Drag Model Used in Ref. 4 (Case 2)

Schultz[4] showed that when using the drag model

$$D=C_{D_0}(V)qS+NL^2/qS \tag{68}$$

where N is a constant, the generalized Legendre-Clebsch condition is satisfied. However, we show here that for the drag model Eq. (68), the cruise arc is a nonminimizing arc. After some algebraic manipulation Eq. (49) becomes

$$Y(\omega)=\frac{\sigma D\bar{K}}{W^2H_{VV}}-\frac{[\sigma^2W^2\omega^2/V^2+\sigma^2(D\beta/V+(C_{D_0})_VqS)^2V^2]/M^2V^2\bar{K}}{[\omega^2-g(H_{VV}\beta+\sigma D_{Vh}/V)/\bar{K}]^2+\sigma^2W^4\omega^2/M^2V^6\bar{K}^2} \tag{69}$$

where

$$\bar{K}=H_{VV}-\sigma D/V^2 \tag{70}$$

The term $H_{VV}\beta+\sigma D_{Vh}/V$ is easily shown to be positive. Again it turns out that Q'_{22} and Q'_{33} defined in Eqs. (36) and used in Eq. (38) are negative. To simplify Eq. (69) let

$$\omega_s^2\triangleq g(H_{VV}\beta+\sigma D_{Vh}/V)/\bar{K}$$

so that

$$Y(\omega_s)<\left(\frac{\sigma D\bar{K}}{W^2H_{VV}}-\frac{\bar{K}V^2}{W^2}\right)=\frac{\bar{K}}{W^2D_{VV}}(D-D_{VV}V^2) \tag{71}$$

Making use of Eq. (58)

$$D_{VV}V^2=(C_{D_0})_{VV}V^2qS+4(C_{D_0})_VVqS+3D \tag{72}$$

Therefore, $Y(\omega_s)<0$ and the cruise arc is not minimizing.

VI. Some Intuitive Rationale for Fuel Efficiency of Nonsteady-State Cruise Arcs

The reason for fuel savings may come about for two distinctly different reasons. First, if the dynamics associated with the altitude and flight path angle are neglected, the energy-height approximation yields an infinitely fast chattering solution in the control space of thrust and velocity, when the cruise condition does not coincide with the minimum drag condition.[11] At constant energy by chattering between where the aircraft is aerodynamically efficient (minimum drag at zero thrust), and where the aircraft is power efficient with respect to thrust, the fuel efficiency of the aircraft is improved over that of the steady-state cruise.

However, the neglected dynamics in the energy-height method may be modulated also to enhance fuel efficiency. If the special case where the cruise and the minimum drag conditions coincide is considered, then according to the above performance argument a cyclic arc will not improve fuel performance over steady-state cruise. If cruise is not minimizing in this case it will be due totally to dynamic considerations. To do this choose C_{D_0} and ζ constant in the drag model Eq. (55) and the specific fuel consumption as $\sigma=\nu V$ where ν is a constant. The cruise altitude and velocity is found to be any point on a constant dynamic pressure line obtained by minimizing the drag model Eq. (55).

To test the optimality of cruise, the same procedure is adopted as given in Sec. III, i.e., the same Goh's transformation and exactly the same transformation of controls given in Eq. (34) is used. The major difference is that since there will remain cross term between the state and control variables in the second variation of the performance index, the general form of the frequency test given by Refs. 9 and 10 must be used. The result is that Eq. (47) is now of the form

$$Y(\omega)=\begin{bmatrix} H_{VV} & 0 \\ 0 & H_{LL}\bar{\omega}^2[\bar{\omega}^2+(1-2b)/b]/(\bar{\omega}^2-1)^2 \end{bmatrix} \tag{73}$$

where

$$\bar{\omega}^2=\omega^2/\beta g, \quad b=4(VD/W)^2\beta/g \tag{74}$$

Note that the steady-state cruise will be nonminimizing if

$$(L/D)_{\max}/V<2\sqrt{2\beta/g} \tag{75}$$

If either the maximum $L/D((L/D)_{\max})$ is low or the velocity is high, a cyclic process will be minimizing. The strategy for increasing fuel efficiency is to properly modulate the interchange of potential and kinetic energy. Furthermore, if $2b$ is only slightly larger than 1, then the frequencies which will give improved fuel performance are quite small indicating long climbs and descents. If $b>1$, then the dominant frequency for improving cruise is $\omega=(g\beta)^{1/2}$ as was found for the case considered in Sec. V.

VII. Conclusions

It is shown that even if the cruise arc satisfies the Euler-Lagrange conditions and the generalized Legendre-Clebsch condition, for a large class of drag models the Jacobi test would fail. In particular, the drag model suggested by Schultz[4] is included in this class. This was done by taking advantage of the stationarity of the second variational problem, and using Parseval's relation to obtain an algebraic frequency condition. In the case of constant specific fuel consumption, this easily used condition states that if a certain scalar function of frequency is negative for any frequency, then the cruise arc is not minimizing. This means that if the cruise arc is long enough, a conjugate point will occur, and the cruise arc thereafter is no longer minimizing. However, before the conjugate point is reached the cruise arc is minimizing.

The results here give little indication as to the form of the improved cyclic optimum path. This is because the optimum path will be a large variation in the state away from the cruise arc, and the theory and results presented are based on making only small variations in the state and control variables.

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